Exercise 66

(a) Prove Theorem 4, part 3.

(b) Prove Theorem 4, part 5.

Solution

Part 1

Assume that f and g are continuous at a and attempt to show that $f+g$ is also continuous at a. Use the definition of continuity.

$$\lim_{x\to a} f(x) = f(a) \qquad \lim_{x\to a} g(x) = g(a)$$

Now take the limit of $f+g$ as $x \to a$, and use the fact that the limit of a sum is the sum of the limits.

$$\begin{aligned}\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\ &= f(a)+g(a) \\ &= (f+g)(a)\end{aligned}$$

Therefore, $f+g$ is continuous at a.

Part 2

Assume that f and g are continuous at a and attempt to show that $f-g$ is also continuous at a. Use the definition of continuity.

$$\lim_{x\to a} f(x) = f(a) \qquad \lim_{x\to a} g(x) = g(a)$$

Now take the limit of $f+g$ as $x \to a$, and use the fact that the limit of a difference is the difference of the limits.

$$\begin{aligned}\lim_{x\to a}(f-g)(x) &= \lim_{x\to a}[f(x)-g(x)] \\ &= \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \\ &= f(a)-g(a) \\ &= (f-g)(a)\end{aligned}$$

Therefore, $f-g$ is continuous at a.

Part 3

Assume that f is continuous at a, that c is a constant, and attempt to show that cf is also continuous at a. Use the definition of continuity.

$$\lim_{x\to a} f(x) = f(a)$$

Now take the limit of cf as $x \to a$, and use the fact that the limit of a constant times a function is the constant times the limit of the function.

$$\begin{aligned}\lim_{x\to a}(cf)(x) &= \lim_{x\to a}[cf(x)] \\ &= c\left[\lim_{x\to a} f(x)\right] \\ &= c[f(a)] \\ &= (cf)(a)\end{aligned}$$

Therefore, cf is continuous at a.

Part 4

Assume that f and g are continuous at a and attempt to show that fg is also continuous at a. Use the definition of continuity.

$$\lim_{x\to a} f(x) = f(a) \qquad \lim_{x\to a} g(x) = g(a)$$

Now take the limit of fg as $x \to a$, and use the fact that the limit of a product is the product of the limits.

$$\begin{aligned}\lim_{x\to a}(fg)(x) &= \lim_{x\to a}[f(x)g(x)] \\ &= \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] \\ &= f(a)g(a) \\ &= (fg)(a)\end{aligned}$$

Therefore, fg is continuous at a.

Part 5

Assume that f and g are continuous at a and attempt to show that f/g is also continuous at a if $g(a) \neq 0$. Use the definition of continuity.

$$\lim_{x\to a} f(x) = f(a) \qquad \lim_{x\to a} g(x) = g(a)$$

Now take the limit of f/g as $x \to a$, and use the fact that the limit of a quotient is the quotient of the limits, provided that the denominator is not zero.

$$\begin{aligned}\lim_{x\to a}\left(\frac{f}{g}\right)(x) &= \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] \\ &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad \lim_{x\to a} g(x) \neq 0 \\ &= \frac{f(a)}{g(a)}, \quad g(a) \neq 0 \\ &= \left(\frac{f}{g}\right)(a), \quad g(a) \neq 0\end{aligned}$$

Therefore, f/g is continuous at a, provided that $g(a) \neq 0$.